Pałasińska's finite basis theorem revisited

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outline

1. Old stuff
   - Finite axiomatizability
   - SUH classes
   - Pałasińska's theorem

2. New stuff - proof
   - filter formulas
   - better universe

theories, classes

L - default finite language
F - fragment of FO logic (set of sentences in L)
M - model

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General Problem

When $\mathbb{Th}_F(M)$, or $F(M)$, is finitely axiomatizable?

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Trivial fact

If M is a finite model, $\mathbb{T}\mathrm{h}(M)$ is finitely axiomatizable.

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Trivial fact

If M is a finite model, $\mathbb{Th}(M)$ is finitely axiomatizable.

Highly nontrivial fact (McKenzie '96)

The question whether for a finite algebra A the equational theory $\mathbb{Th}_{Eq}(A)$ is finitely axiomatizable is undecidable.

equality is redundant

Θ - binary predicate symbol not in default language L

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M_Θ model in $\mathrm{L} \cup \{\Theta\}$ s.t.

- its L-reduct is M
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$\mathbb{Th}_F(M)$ is finitely axiomatizable
iff
$\mathbb{Th}_{F \cap \not\approx}(M_\Theta)$ is finitely axiomatizable

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Attention

In what follows we assume that $\approx \notin \mathrm{L}$

strict universal Horn classes

A SUH sentence looks like

$$(\forall \bar{x}) \left[\bigwedge_{i \leqslant n} \varphi_i(\bar{x}) \right] \to \varphi(\bar{x})$$

φ_i, φ - atomic formulas

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A SUH sentence looks like

$$(\forall \bar{x}) \left[\bigwedge_{i \leqslant n} R_i(\bar{t}_i(\bar{x})) \right] \to R(\bar{t}(\bar{x}))$$

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$\mathcal{H} = \{(A, \Theta) \mid A$ - algebra, Θ - congruence of $A\}$

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Axioms of $\mathcal{H}$:
$(\forall x, y, z)[\Theta(x, y) \wedge \Theta(y, z)] \to \Theta(x, z)$

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$(\forall x, y, z)[\Theta(x, y) \wedge \Theta(y, z)] \rightarrow \Theta(x, z)$

$(\forall x, y, u, v)[\Theta(x, y) \wedge \Theta(u, v)] \rightarrow \Theta(\omega(x, u), \omega(y, v))$

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alternatively,
DS $\simeq$ consequence relation $\vdash \subseteq \mathcal{P}(Sent) \times Sent$ satisfying . . . . . .

deductive systems $\leftrightsquigarrow$ SUH classes

Correspondence

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$$\vdash \quad \leftrightsquigarrow \quad \mathcal{H}_{\vdash}$$

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$\vdash$	$\leftrightsquigarrow$	$\mathcal{H}_{\vdash}$
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Fact

A deductive system $\vdash$ may be described by a finite set of axioms and a finite set of inference rules iff $\mathcal{H}_{\vdash}$ is finitely axiomatizable.

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Fact

$\mathcal{Q}$ is finitely axiomatizable iff $\mathcal{Q}_\Theta$ is finitely axiomatizable

protoalgebraicity

Definition

M - model

$\Omega(M)$ - Leibniz congruence of M given by

$(a, b) \in \Omega(M)$ iff $M \models (\forall \bar{z})[\varphi(a, \bar{z}) \leftrightarrow \varphi(b, \bar{z})]$
for every (atomic) formula φ

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Definition

A SUH class $\mathcal{H}$ is protoalgebraic provided for $(A, R), (A, S) \in \mathcal{H}$
if $R \subseteq S$, then $\Omega(A, R) \subseteq \Omega(A, S)$.

protoalgebraicity continued

Example

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Important fact

For a SUH class $\mathcal{H}$ we may define $\mathcal{H}$-subdirectly irreducible models.
If $\mathcal{H}$ is protoalgebraic, $\mathcal{H}$-subdirectly irreducible models behave like relative subdirectly irreducible algebras in a quasivariety.

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Example

A SUH class defined by universal atomic sentences is filter distributive.

Pałasińska's theorem

Pałasińska's theorem '94

$\mathcal{K}$ - finite family of finite models
If $SUH(\mathcal{K})$

- is protoalgebraic and
- is filter-distributive,

then it is finitely axiomatizable.

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Corollary I

A finitely generated protoalgebraic filter distributive deductive system may be described by a finite set of axioms and a finite set of inference rules.

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Corollary I

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Corollary II: Pigozzi's theorem '88

A finitely generated relatively congruence-distributive quasivariety is finitely axiomatizable.

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- analog of congruence formulas for varieties and relative congruence formulas for quasivarieties.

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Definition

$\mathcal{H}$-filter formula $\Gamma(y, x)$ looks like

$$(\exists \bar{z}) \bigwedge [t_i(x, y, \bar{z}) \approx s_i(x, y, \bar{z})] \wedge \bigwedge R(r_j(x, y, \bar{z}))$$

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But it does not work. We cannot use $\approx$.

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where $\sim$ is Leibniz congruence.

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finitely axiomatizable protoalgebraic SUH class

Proposition

Let $\mathcal{H}$ be a protoalgebraic SUH class such that its subdirectly irreducible members form the axiomatizable class. Then there is a SUH class $\mathcal{U}$ such that

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and we may define filter formulas within $\mathcal{U}$.

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The end :-)